A degradation-based replacement policy for complex multi-component systems

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Given a reward structure, this paper addresses an optimal replacement problem for complex multi-component systems. To maintain revenue stream resulting from system, the system is inspected at random times and certain actions are performed in response to the system state. Decisions are based on a performance measure described by a Squared Bessel process. Since there are some flow of income and increasing costs due to inspections, the problem is to optimally stop processing the system and carrying out a renewal to maximize the reward functional. In support of the model a numerical example is provided to demonstrate the application of the proposed model.

Keywords: Replacement; Inspection; Multi-Component System; Squared Bessel process; Extended Gamma process

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1 Introduction

Consider a revenue generating gas turbine consisting of n components. It is assumed that the system state is described by an n-dimensional Wiener process $\mathbf{W}_t$. Decisions are based on a performance measure, the squared norm of $\mathbf{W}_t$, which describes the total degradation of components. The arguments on the performance measure are developed with some restriction on the process. This allows a modified form of the performance measure that is an extended Gamma process is explored. It is shown that the extended Gamma process treated as a Gamma process with stochastic shape parameter and time-dependent scale parameter is a mixture of stochastic motions and random jumps which respectively conform to the Gamma process and the non-homogeneous Poisson process. For condition monitoring and so decision making, the system is inspected according to a homogeneous Poisson process and certain actions are carried out in response to the observed system state. Each inspection at time t incurs the cost $C(t) = Ct$ $(C > 0)$. On the other hand, there is a discounted flow of reward from system. The discounted factor allows future reward to be discounted, a case most common in real application. Using the local characteristic of EGP as the stopping time of the process, we get an estimate of the net profit over a cycle that is the difference of the discounted reward resulting from system and the maintenance costs. This modeling of replacement is similar to those [7, 9] using accumulated damage or total degradation as a basis for decision making. In many other studies in the literature [4], the replacement of a system is modelled as an N-policy where an immediate replacement is carried out as number of shocks reaches N. Since the stopping time controls both the amount of maintenance and the reward, the problem is to determine an optimal operating time (production run length) which truly balances the flow of reward and increasing cost due to inspections. Due to some distinctive characteristics of the EGP, specifically, the degradation physics of the EGP described above, the approach presented is typically appropriate for systems such as gas turbines whose degradation phenomena is considered as accumulations of additive and irreversible damage caused by environment factors and stress. Some examples with above characteristics are corrosion, static and alternating stress, thermal stress and physical wear. We will see that our maintenance modelling technique benefiting from the local characteristic of the EGP provides more relaxed and generalized approach to the maintenance scheduling problem with some characteristics which have not been addressed or previously studied in isolation. Basically, using a multivariate system model whose aggregate measure of performance is the EGP, we develop maintenance models whose attention is either restricted to perfect repair/inspection, or turned to maintaining single unit systems [5, 6, 10].

2 Modelling the degradation

Consider the revenue generating gas turbine that deteriorates over time due to normal usage and aging. The turbine state is characterized by two states: "in-control" (steady) state, an "out-of-control" state. The model is based on realistic assumptions that operation in either one of the states generates income, which is higher in the in-control state. The assumption implies that a less degraded state of the system results in better performance of the system and so more income from system. For degradation modeling, we benefit from an aggregate performance measure that is the squared norm of n-dimensional Wiener process. For this, let the state of n components be described by an n-dimensional Wiener process with drift parameter μ_i and variance parameter σ^2:

$$W_i(t) = \mu_i t + \sigma B_i(t) \quad i = 1, 2, \cdots n, \tag{1}$$

where $B_i(t)$ is a standard Brownian motion. The state of the system over time is described by an aggregate performance measure $D(t;x)$ that is the squared norm of $\mathbf{W_t}$ starting from state x. In other words, if

$$\mathbf{W_t} = [W_1(t), W_2(t), \cdots, W_n(t)] = \underline{\mu} t + \mathbf{B}(t),$$

then

$$D(t;x) = \|\mathbf{W_t}\|_2 = \sum_{i=1}^{n} W_t^2(t), \tag{2}$$

where $\underline{\mu}$ and $\mathbf{B}(t)$ are the vector of drift parameters and standard Brownian motions respectively. The assumption of the common variance allows to get the Laplace transform $\phi(x, \underline{\vartheta})$ of the squared norm $D(t;x)$ conforming to an n-dimensional squared Bessel process [8]:

$$\phi(x, \underline{\vartheta}) = \mathbb{E}\left(e^{-\lambda D(t;x)}\right) = \left(1 + 2\lambda\sigma^2 t\right)^{\frac{-n}{2}} \times \exp\left(\frac{-\lambda}{1 + 2\lambda\sigma^2 t} \sum_{i=1}^{n} (\mu_i t + x)^2\right), \tag{3}$$

where λ is a real number, $\underline{\vartheta} = (\upsilon, \mu(t;x))$, υ and μ are the parameter and the drift term:

$$\upsilon = \frac{n}{2} - 1, \qquad \mu(t;x) = \sum_{i=1}^{n} (\mu_i t + x)^2.$$

The Laplace transform (3) enable us to get the expected value of the squared Bessel process $D(t;x)$ that is

$$\mathbb{E}\left(D(t;x)\right) = n\sigma^2 t + \mu(t;x).$$

In the following proposition we show that given some assumptions, the squared Bessel process (2) can be represented as an extended Gamma process.

Proposition 2.1 *Let $D(t;x)$ be the squared Bessel process* (2)*:*

$$D(t;x) \sim \chi^2_{\underline{\vartheta}}$$

Given that the starting state of the system is $x = 0$ and the number of components are even $(n = 2k)$ $(k \in \mathbb{N})$, the squared Bessel process $D_t \equiv D(t;0)$ is an extended Gamma process with the stochastic shape parameter $\alpha(t) = N(t;\theta) + \frac{n}{2}$ and the scale parameter $\beta(t) = 2\sigma^2 t$. That is,

$$D_t \overset{d}{=} Z_t \sim \mathbf{Gamma}\left(\alpha(t), \beta(t)\right)$$

where $N(t;\theta)$ conforms to a non-homogeneous Poisson process with the time point process (S_k) $(k \geq 1)$. In other words, $N(t;\theta)$ admits a smooth semi-martingale representation $N = (\lambda, \mathbb{M})$ [2]:

$$\begin{aligned} N(t;\theta) &= \int_0^t \lambda(s;\theta)ds + \mathbb{M}_t \\ &= \Lambda_\theta(t) + \mathbb{M}_t \end{aligned} \tag{4}$$

where θ is the vector of parameters, $\theta = \left(\underline{\mu}, \sigma^2\right)$,

$$\lambda(t;\theta) = \frac{1}{2}\sum_{i=1}^{n}\frac{\mu_i^2}{\sigma^2}t,$$

and $\mathbb{M}_t$ denote the intensity and the martingale adapted to the general filtration $\mathcal{F}_t$.
Proof: *A proof of the Proposition 2.1 is given by Ahmadi [1].* □

3 Maintenance decision mechanism

To maintain revenue stream from system, the system is inspected according a homogeneous Poisson process $N(t;\gamma)$ with the arrival rate $\lambda(t;\gamma) = \gamma t(\gamma > 0)$ and certain actions are performed in response to the system state. Decisions at inspection times are based on the local characteristic of the performance measure Z_t partitioning the state space Ω into an in-control state G and an out-of-control state G^c. In other words, the process is stopped and reset back to initial state if the total degradation of the components described by the performance measure Z_t exceeds the degradation threshold $Z_k \equiv Z_{S_k}$ at terminal time S_k $(k \in \mathbb{N})$, the first entry into the out-of-control state $G^c = [Z_k, \infty)$:

$$S_k = \inf\left\{t : N(t;\theta) = k\right\}, \tag{5}$$

and the system operates if $Z_t \in G = [0, Z_k)$ (in-control state). At hitting time S_k, a replacement is instantaneously performed at cost K and the renewal occurs. As noted from (5), the replacement of the system is determined by the first hitting time (or, departure time of the process from in-control state to out-of-control state) upon which the total number of jump arrivals reaches k. So, the replacement rule can be considered as threshold-type policy or k-replacement policy. In the sense that the system is replaced by new one if the total degradation of components reaches the critical level Z_k, or equivalently, the total number of jumps conforming to the NPP (4) reaches k. Our aim is to get an optimal operating time determined by the optimal number of arrivals k^* which balances some flow of reward resulting from system and increasing inspection costs rate due to inspections. This leads to maximizing discounted reward of the proposed policy. Sections 4 and 5 show the feasibility of this programme.
The distribution function $F_k(t)$ of the k^{th} arrival time (stoping time) in the NPP with arrival rate $\lambda(t;\theta)$, S_k, is given by

$$\begin{aligned} F_k(t) = \mathbf{P}(N(t;\theta) \geq k) &= \sum_{i=k}^{\infty}\frac{\exp\left(-\Lambda_\theta(t)\right)\Lambda_\theta(t)^i}{i!}, \\ &= 1 - \exp\left(-\Lambda_\theta(t)\right) \times \sum_{i=0}^{k-1}\frac{\Lambda_\theta(t)^i}{i!} \end{aligned} \tag{6}$$

with the corresponding density function $f_k(t)$:

$$f_k(t) = \lambda(t;\theta) \times \frac{\exp\left(-\Lambda_\theta(t)\right)\Lambda_\theta(t)^{k-1}}{(k-1)!}, \tag{7}$$

the hitting time intensity function $\lambda_k(t;\theta)$:

$$\lambda_k(t;\theta) = \frac{f_k(t)}{\bar{F}_k(t)},$$

and the mean hitting time of the critical time G^c:

$$\mu_k = \int_0^\infty \bar{F}_k(t)dt. \tag{8}$$

From equation (6) it is easy to see that the cumulative density function of the defined stopping time (5) caused by performance measure passing a degradation threshold at k^{th} jump is explicit. This distinctive feature underpins the

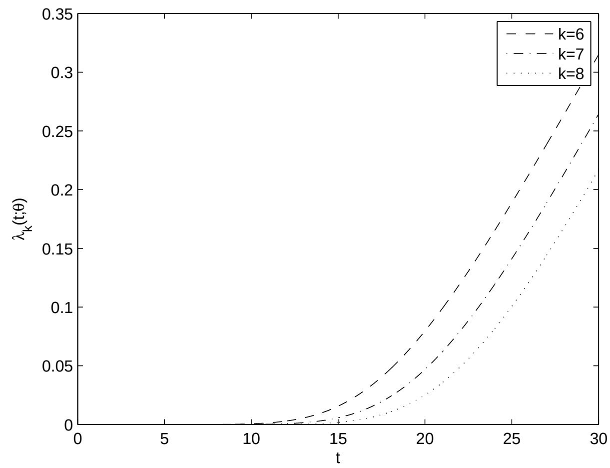

Fig. 1: Hitting time intensity $\lambda_k(t;\theta) = f_k(t)/\bar{F}_k(t)$ for $k = 6, 7, 8$ given $(\mu_i, \sigma^2) = (0.1, 1)$ and $n = 4$.

extended Gamma process as more suitable degradation model in contrast to the squared Bessel process whose first passage time distribution due to its implicit form does not analytically accommodate an exact statistical inference. The Laplace Transform (LT) of the PDF of the first passage time S_k $(k \geq 1)$ can be given by

$$F(\lambda) = \mathcal{L}\left[f_k(t)\right] = \sum_{u=0}^{\infty} \frac{\left[\alpha(\underline{\mu}, \sigma)\right]^u}{u!} \times \frac{\Gamma(\frac{u}{2} + k)}{\Gamma(k)} \tag{9}$$

where $\lambda(\underline{\mu}, \sigma) = -2\sigma\lambda/\sqrt{\sum_{i=1}^{n} \mu_i^2}$. Using the LT of S_k, we get the mean hitting time of the critical set G^c, that is

$$\mu_k = \mathbb{E}(S_k) = \frac{2\sigma}{\sum_{i=1}^{n} \mu_i^2} \times \frac{\Gamma(\frac{1}{2} + k)}{\Gamma(k)}.$$

In support of above argument, the response of the maintenance model to the number of arrivals k is examined for different k (see Figures 1). As illustrated changing k affects the hitting time distribution $F_k(t)$ and the hitting time intensity $\lambda_k(t;\theta)$ so that decreasing k rises the hitting time probability of the critical set G^c.

4 Reward model

Using $\alpha \geq 0$ as a continuous discount factor, the expected total discounted reward over $[0, S_k]$ starting in state $Z_0 = 0$ is given by

$$\mathcal{C}_\alpha = \mathbb{E}\left[\int_0^{S_k} e^{-\alpha s} Z_s ds - \int_0^{S_k} C(s) dN(s;\gamma) - K\right] \tag{10}$$

Since the counting process $N(t;\gamma)$ has the $\mathcal{F}$-intensity $\lambda(t;\gamma)$ and $C(t) = Ct$ is a $\mathcal{F}$-predictable process, using the argument given in [3], one can show that

$$\begin{aligned} \mathcal{C}_\alpha &= \mathbb{E}\left[\int_0^{S_k} \left(e^{-\alpha s} Z_s - C(s)\lambda(s;\gamma)\right) ds - K\right] \\ &= \mathcal{C}_\alpha^{(1)} - \mathcal{C}_\alpha^{(2)} - \mathcal{C}_\alpha^{(3)} \end{aligned} \tag{11}$$

which includes the expected discounted reward $\mathcal{C}_\alpha^{(1)}$ earned from system at rate $e^{-\alpha t} Z_t$ $(t > 0)$, the expected inspection cost $\mathcal{C}_\alpha^{(2)}$ over the cycle $[0, S_k]$ and the replacement cost $\mathcal{C}_\alpha^{(3)} = K$ incurred at stopping time S_k:

$$\mathcal{C}_\alpha^{(1)} = C_0(\alpha) + C_1(\alpha)\mathcal{L}\left[f_k(t)\right] + C_2(\alpha)\mathcal{L}\left[f_k(t)/t\right] + C_3(\alpha)\mathcal{L}\left[f_k(t)/t^2\right] + C_4(\alpha)\frac{d}{d\alpha}\mathcal{L}\left[f_k(t)\right], \tag{12}$$

that for $k > 1$

$$C_0(\alpha) = \frac{3k\sum_{i=1}^{n}\mu_i^2}{(k-1)\alpha^4} + \frac{n\sigma^2}{\alpha^2}, \quad C_1(\alpha) = \frac{-(n+6k)\sigma^2}{\alpha^2}, \quad C_2(\alpha) = \frac{-12k\sigma^2}{\alpha^3},$$

$$C_3(\alpha) = \frac{1}{\alpha}C_2(\alpha), \quad C_4(\alpha) = \frac{(n+2k)\sigma^2}{\alpha}.$$

To get the expected total inspection cost over $[0, S_k]$, let inspections occur according to a HPP with arrival rate $\lambda(t;\gamma) = \gamma t$ and each inspection at time t $(t \geq 0)$ incurs a cost $C(t) = Ct$ $(C > 0)$. The same argument as above follows that

$$\mathcal{C}_\alpha^{(2)} = \mathbb{E}\left[\int_0^{S_k} C(s)\lambda(s;\gamma)ds\right] = -\frac{C\lambda_\gamma}{3}F^{(3)}(\alpha)|_{\alpha=0},$$

where $F^{(i)}(\alpha)$ $i = 1, 2, 3, \cdots$ denotes the derivative of $F(\alpha) = \mathcal{L}[f_k(t)]$ of order $i \in \mathbb{N}$ with respect to α:

$$F^{(i)}(\alpha) = \frac{d^i}{d\alpha^i}\mathcal{L}\left[f_k(t)\right] = \left(\frac{-2\sigma}{\sqrt{\sum_{i=1}^{n}\mu_i^2}}\right)^i \times \sum_{u=0}^{\infty}\left(\frac{\left[\alpha(\underline{\mu}, \sigma)\right]^u}{u!} \times \frac{\Gamma\left(\frac{u+i}{2}+k\right)}{\Gamma(k)}\right).$$

Thus, the expected total discounted reward over $[0, S_k]$ becomes

$$\mathcal{C}_\alpha = \mathcal{C}_\alpha^{(1)} - \mathcal{C}_\alpha^{(2)} - K.$$

5 Optimizing the model

Our aim is to obtain an optimal replacement policy determined by the optimal number of jumps "k" which maximizes the expected total discounted reward of the proposed policy. To this end, let the cycle begin with the system in the as-good-as-new (stable) condition. Assume that the discount factor is $\alpha = 0.2$. The choice for costs and the degradation and maintenance function's parameters are $(C, K) = \left(9 \times 10^{-4}, 2\right)$, $(\mu_i, \sigma^2) = (0.1, 1)$ and $\gamma = 1$ respectively. From Figure 2 it is easy to see that the expected total discounted reward $\mathcal{C}_\alpha$ is concave in the number of external jumps and as indicated $\mathcal{C}_\alpha$ achieves its maximum at 12^{th} external jump ($k_\alpha^* = 12$). By plugging the optimal solution into the equation (8) an optimal solution to the optimal operating time (production run length) of the system that is $\mu_{k_\alpha^*} = 171.41$ is derived. Therefore, to stop shifting the process from in-control state to out-of control state and so maintaining the reward stream from the system, it is suggested the process should be stopped at 12^{th} jump and the replacement is scheduled at optimal time $\mu_{k_\alpha^*} = 171.41$. The proposed policy assures keeping the process (performance measure) in-control state, the life-cycle profitable and so rising revenue from the system with the optimum reward $\mathcal{C}_\alpha^* = 155.522$.

6 Conclusion

Given a reward structure, this paper provides an approach to the determination of an optimal replacement policy for a complex multi-component system (e.g. gas turbine) whose state is described by an extended Gamma process. Explored as the modified form of the squared Bessel process, the extended Gamma process and its local characteristic incorporate into the reward model to get an estimate of the net profit as the difference of the reward resulting from system and the maintenance costs. The expected total discounted reward is optimized over a cycle with a terminal time defined as the k^{th} arrival time of a NPP. The optimal number of arrival k^* determines the optimal operating time which balances some flow of reward resulting from system and the increasing cost rate due to inspections. The results of the model provide sensible and realistic replacement policy for such systems. The paper outlined an approach which can be extended later by implementing degradation and imperfect modeling techniques for monotone complex systems.

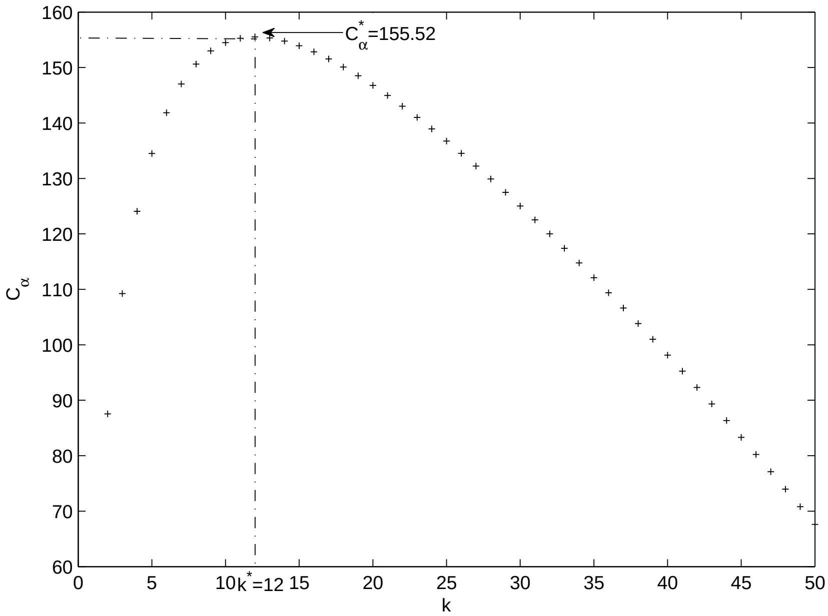

Fig. 2: Expected total discounted reward as a function of the number of arrivals k given $(\mu_i, \sigma^2) = (0.1, 1)$, $n = 4$, $\gamma = 1$, the discounted factor $\alpha = 0.2$ and the inspection cost per unit time $C = 9 \times 10^{-4}$ and replacement cost $K = 2$.

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