

Modeling a degradation process with dependent increments in presence of random heterogeneity

Massimiliano Giorgio¹, Maurizio Guida^{2,3} and Gianpaolo Pulcini³

¹ *Department of Industrial and Information Engineering, Second University of Napoli, 81031 Aversa (CE), Italy, massimiliano.giorgio@unina2.it*

² *Department of Information Engineering, Electrical Engineering and Applied Mathematics, University of Salerno, 84084 Fisciano (SA), Italy, mguida@unisa.it*

³ *Istituto Motori, National Research Council (CNR), 80125 Napoli, Italy, g.pulcini@im.cnr.it*

This paper aims to model the degradation paths of degrading units in presence of an unexplained form of heterogeneity among the paths. Focus is on monotonic increasing degradation processes where the degradation increments over disjoint time intervals are not independent. The degradation path of each unit is described via the Transformed Gamma process, and the “age” and “state” functions that characterize the Transformed Gamma process are here assumed to be power-law functions. The unexplained heterogeneity among paths of different units is accounted for assuming that the scale parameters of the “age” and “state” functions vary from unit to unit. This variability is modeled assuming that the scale parameters are independent gamma random variables. Under these assumptions, a quite mathematically tractable model is obtained. The main properties of the proposed model are discussed, and inferential procedures based on the maximum likelihood criterion are presented. Finally, the proposed model is applied to a real set of degradation data to show the feasibility of the proposed model.

Keywords: Degradation processes, Transformed Gamma process, dependent increments, random effects, maximum likelihood estimate

1 Introduction

This paper aims to model the degradation paths of degrading units in presence of an unexplained form of heterogeneity among the paths. Focus is on monotonic increasing degradation processes $\{W(t); t \geq 0\}$ where the degradation increments over disjoint time intervals are not independent. The degradation path of each unit is described via the Transformed Gamma (TG) process, a Markovian degradation process that was initially proposed in [1]. The TG process is a very flexible age- and state-dependent degradation process that possesses the distinguishing features of being mathematically tractable; for example, the conditional probability density function (pdf) of the degradation growth over a future time interval, given the current degradation level, is in closed form.

In addition, since the TG process can be viewed as a generalization of the gamma (G) process, obtained by a non-linear transformation of the gamma process, the applicative contexts of the TG process include those of the gamma process, specifically degradation phenomena where degradation growth takes place gradually over time in a sequence of tiny increments [2]–[3]. Thus, the TG process seems to be suitable to describe degradation phenomena caused by continuous use, such as wear, chemical corrosion, consumption, fatigue, and so on.

Heterogeneity among units is possibly due to internal and/or external factors. In case of observable factors, heterogeneity can be modeled by conditioning the stochastic description to a vector of covariates. Recent examples of this approach can be found in [1] and [4], where the case of time-independent and time-dependent covariates are considered, respectively. In many cases, however, heterogeneity is due to some unobservable

factors. In such a case, the previous approach is no more applicable, and the alternative is to assume that some (or all) of the stochastic model parameters are random variables.

Thus, the aim of this paper is to extend the TG process by incorporating random effects. In particular, we assume that the “age” and “state” functions that characterize the TG process are both power-law functions, say $\eta(t) = a t^b$ and $g(w) = \alpha w^\beta$, respectively, and that the unexplained heterogeneity among the paths of different units is accounted for by assuming that the scale parameters a and α of the “age” and “state” functions, respectively, vary randomly from unit to unit. In particular, both the scale parameters are assumed to be independent gamma random variables. Under such an assumption, a quite mathematically tractable model is obtained, since the (conditional) probability distribution of the degradation increment $\Delta W(t, t+\tau)$ over the future time interval $(t, t+\tau)$ involves only univariate integrations.

The main properties of the proposed model are discussed. In particular, it is shown that when the scale parameter of the “age” function is common across the units, so that the heterogeneity involves only the “state” function, the pdf of $\Delta W(t, t+\tau)$ is in closed form. Inferential procedures based on the maximum likelihood criterion are also presented.

Finally, the proposed model is applied to a real set of degradation data, consisting in the light intensity of a heterogeneous sample of LEDs put on test under a constant level of electric current, in order to show the feasibility of the proposed model and estimation procedure.

2 The transformed gamma process with random effects

Let $\eta(t)$ be a non-negative, monotone increasing function of time t , in the following called “age function”, with $\eta(0)=0$, and let $g(w)$ be a non-negative, monotone increasing and differentiable function of the degradation level w , in the following called “state function”, with $g(0)=0$. An increasing degradation process $\{W(t); t \geq 0\}$ is said to be a TG process with age function $\eta(t)$ and state function $g(w)$ if it possesses the following properties:

1. the degradation increments over disjoint time intervals are not possibly independent,
2. the degradation increment $\Delta W(t, \tau) \equiv W(t+\tau) - W(t)$ over the time interval $(t, t+\tau)$ depends on the process history up to time t , say H_t , through the current time t and the current state (degradation level) $w_t = W(t)$, only, being independent on the past,
3. the (conditional) pdf of $\Delta W(t, \tau)$ is given by:

$$f_{\Delta W(t, \tau)}(\delta | w_t) = g'(w_t + \delta) \frac{[g(w_t, \delta)]^{\eta(t, \tau)-1}}{\Gamma[\eta(t, \tau)]} \exp[-g(w_t, \delta)] \quad , \quad \delta > 0 \quad , \quad (1)$$

where $g'(w_t + \delta)$ is the first derivative of the state function $g(w)$ evaluated at $w_t + \delta$, $g(w_t, \delta) = g(w_t + \delta) - g(w_t)$, $\eta(t, \tau) = \eta(t+\tau) - \eta(t)$, and $\Gamma(\bullet)$ is the complete gamma function.

If $\eta(t)$ is linear with the age t , then the (conditional) distribution of the increment $\Delta W(t, \tau)$ depends on the interval width τ and not on the current age t , so that the TG process is said to be age-independent. On the other side, if $g(w)$ is linear with w , the distribution of $\Delta W(t, \tau)$ is independent of the current level w_t , and the TG process reduces to a (state-independent) gamma process with shape function $\eta(t)$ and fixed scale parameter.

From (1), the pdf and the Cdf of the degradation level $W(t)$ at the time t of a new (unused) unit, in absence of heterogeneity among the units, are respectively given by:

$$f_{W(t)}(w) = g'(w) \frac{[g(w)]^{\eta(t)-1}}{\Gamma[\eta(t)]} \exp[-g(w)] \quad , \quad (2)$$

$$F_{W(t)}(w) = \frac{\text{IG}[g(w); \eta(t)]}{\Gamma[\eta(t)]} \quad , \quad (3)$$

where $\text{IG}[y; s]$ is the (lower) incomplete gamma function.

Several functional forms for the age and state functions can be chosen, as discussed in [1] and [5]. Following [5], a power-law function is used for both $\eta(t)$ and $g(w)$, say:

$$\eta(t) = a t^b \quad \text{and} \quad g(w) = \alpha w^\beta. \quad (4)$$

Under such formulation, the TG process becomes age-independent when $b=1$, and is state-independent when $\beta=1$. The mean and variance of the degradation level $W(t)$, in absence of heterogeneity, are in closed form, and given by:

$$E\{W(t)\} = \frac{1}{\alpha^{1/\beta}} \frac{\Gamma(a t^b + 1/\beta)}{\Gamma(a t^b)} \quad \text{and} \quad V\{W(t)\} = \frac{1}{\alpha^{2/\beta}} \left(\frac{\Gamma(a t^b + 2/\beta)}{\Gamma(a t^b)} - \frac{\Gamma^2(a t^b + 1/\beta)}{\Gamma^2(a t^b)} \right). \quad (5)$$

To describe the random heterogeneity among the units, we extend the assumption made in [6] for the gamma process where the heterogeneity has effect only on the scale parameter of the gamma process, and hence we assume that the shape parameters b and β are fixed and that the scale parameters a and α are random variables. Thus, the conditional pdf of the degradation increment $\Delta W(t, t+\tau)$ over the time interval $(t, t+\tau)$, given the unknown (and random) parameters a and α is:

$$f_{\Delta W(t, \tau)}(\delta | w_t, a, \alpha) = \beta \alpha (w_t + \delta)^{\beta-1} \frac{\{\alpha[(w_t + \delta)^\beta - w_t^\beta]\}^{a[(t+\tau)^b - t^b] - 1}}{\Gamma\{a[(t+\tau)^b - t^b]\}} \exp\{-\alpha[(w_t + \delta)^\beta - w_t^\beta]\}. \quad (6)$$

In addition, we assume that a and α are independent gamma distributed random variables with parameters (r, s) and (c, d) , respectively:

$$f(a) = \frac{s^r a^{r-1}}{\Gamma(r)} \exp(-sa), \quad a > 0, \quad r, s > 0 \quad (7)$$

$$f(\alpha) = \frac{d^c \alpha^{c-1}}{\Gamma(c)} \exp(-d\alpha), \quad \alpha > 0, \quad c, d > 0 \quad (8)$$

so that $E\{a\} = r/s$, $V\{a\} = r/s^2$, $E\{\alpha\} = c/d$, and $V\{\alpha\} = c/d^2$. Under the above assumption on the distribution of a and α , the mean and variance of the degradation level $W(t)$ relative to the heterogeneous population of units result in:

$$E\{W(t)\} = d^{1/\beta} \frac{s^r}{\Gamma(r)} \frac{\Gamma(c-1/\beta)}{\Gamma(c)} \int_0^\infty \frac{\Gamma(a t^b + 1/\beta)}{\Gamma(a t^b)} a^{r-1} \exp(-sa) da \quad (9)$$

$$V\{W(t)\} = d^{2/\beta} \frac{s^r}{\Gamma(r)} \frac{\Gamma(c-2/\beta)}{\Gamma(c)} \int_0^\infty \frac{\Gamma(a t^b + 2/\beta)}{\Gamma(a t^b)} a^{r-1} \exp(-sa) da - E^2\{W(t)\}. \quad (10)$$

It should be noted that, differently from the gamma process, the variance of the TG process is not constrained to increase monotonically, both in absence [1] and in presence of random effects.

3 Maximum likelihood estimation and prediction

Let us suppose that m units subject to degradation phenomena are observed, and that random heterogeneity exists among the units and/or their operational conditions. Each unit is inspected n_i times at possibly not equal ages $t_{i,j}$ ($i=1, \dots, m$ and $j=1, \dots, n_i$). Let $w_{i,j} = W(t_{i,j})$ denote the degradation level of the unit i measured at the j -th inspection time $t_{i,j}$. Then, under the assumptions that the degradation processes are TG with $\eta(t) = a t^b$ and $g(w) = \alpha w^\beta$, and that the heterogeneity is modeled by assuming that the scale parameters a and α are random variables, the (conditional) likelihood function relative to the unit i , given a and α , is:

$$L_i(\mathbf{w}_i | a, \alpha) = \beta^{n_i} \left(\prod_{j=1}^{n_i} w_{i,j}^{\beta-1} \right) \frac{\alpha^{\eta(t_{i,n_i})} \prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j-1}}}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})} \exp(-\alpha w_{i,n_i}^\beta), \quad i=1, \dots, m, \quad (11)$$

where $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,n_i})$ is the vector of the observed data of the unit i , $t_{i,0} = w_{i,0} = 0$ for all i , and

$$\Delta\eta_{i,j} = \eta(t_{i,j}) - \eta(t_{i,j-1}) = a(t_{i,j}^b - t_{i,j-1}^b).$$

Thus, by using (6), (7), and (8), the (unconditional) likelihood function relative to unit i is:

$$\begin{aligned} L_i(\mathbf{w}_i) &= \int_0^\infty \int_0^\infty L_i(\mathbf{w}_i | a, \alpha) f(a) f(\alpha) da d\alpha \\ &= \beta^{n_i} \frac{d^c}{\Gamma(c)} \frac{s^r}{\Gamma(r)} \left(\prod_{j=1}^{n_i} w_{i,j}^{\beta-1} \right) \int_0^\infty \frac{a^{r-1} \prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j-1}}}{(w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i})+c}} \frac{\Gamma[\eta(t_{i,n_i})+c]}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})} \exp(-sa) da, \quad (12) \end{aligned}$$

where $\sum_{j=1}^{n_i} \Delta\eta_{i,j} = \eta(t_{i,n_i}) = at_{i,n_i}^b$.

The evaluation of the likelihood function (12) requires a univariate numerical integration. If only the parameter α is random, then the likelihood function $L_i(\mathbf{w}_i)$ is in closed form:

$$L_i(\mathbf{w}_i) = \frac{\beta^{n_i} d^c}{\Gamma(c)} \left(\prod_{j=1}^{n_i} w_{i,j}^{\beta-1} \right) \frac{\prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j-1}}}{(w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i})+c}} \frac{\Gamma[\eta(t_{i,n_i})+c]}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})}, \quad (13)$$

whereas if a is the only random parameter, the likelihood function $L_i(\mathbf{w}_i)$ still requires a numerical integration:

$$L_i(\mathbf{w}_i) = \beta^{n_i} \frac{s^r}{\Gamma(r)} \left(\prod_{j=1}^{n_i} w_{i,j}^{\beta-1} \right) \int_0^\infty \frac{\alpha^{\eta(t_{i,n_i})} a^{r-1} \prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j-1}}}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})} \exp(-\alpha w_{i,n_i}^\beta) \exp(-sa) da. \quad (14)$$

Of course, the likelihood function relative to the whole data set $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_m)$ is:

$$L(\mathbf{w}) = \prod_{i=1}^m L_i(\mathbf{w}_i).$$

Maximum likelihood (ML) estimates of the model parameters can be obtained by using a numerical optimization procedure.

In order to make prediction on the degradation increment during the future time interval $(t_{i,n_i}, t_{i,n_i} + \tau)$ of a generic unit i , by following [6], we first obtain the distribution of the random parameters a and α relative to the (heterogeneous) subpopulation of units that possess the same history of unit i , that is, the units whose degradation levels at the inspection times t_1, \dots, t_{n_i} are equal to $W(t_1) = w_{i,1}, \dots, W(t_{n_i}) = w_{i,n_i}$. By using the theorem of conditional probabilities, we have that

$$\begin{aligned} f(a, \alpha | \mathbf{w}_i) &= \frac{L_i(\mathbf{w}_i | a, \alpha) f(a) f(\alpha)}{\int_0^\infty \int_0^\infty L_i(\mathbf{w}_i | a, \alpha) f(a) f(\alpha) da d\alpha} \\ &= \frac{\alpha^{\eta(t_{i,n_i})+c-1} a^{r-1} \prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j-1}} \exp[-\alpha(w_{i,n_i}^\beta + d)] \exp(-sa)}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j}) \cdot \int_0^\infty \frac{a^{r-1} \prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j-1}}}{(w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i})+c}} \frac{\Gamma[\eta(t_{i,n_i})+c]}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})} \exp(-sa) da} \quad (15) \end{aligned}$$

Thus, by using (15) together to (6), the pdf of the degradation increment of unit i in the future time interval $(t_{i,n_i}, t_{i,n_i} + \tau)$, is given by:

$$\begin{aligned}
f_{\Delta W}(t_{i,n_i}, \tau)(\delta | \mathbf{w}_i) &= \int_0^\infty \int_0^\infty f_{\Delta W}(t_{i,n_i}, \tau)(\delta | w_{i,n_i}, a, \alpha) f(a, \alpha | \mathbf{w}_i) da d\alpha \\
&= \frac{\beta w_{i,n_i+1}^{\beta-1} \int_0^\infty \frac{a^{r-1} \prod_{j=1}^{n_i+1} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j}-1}}{(w_{i,n_i+1}^\beta + d)^{\eta(t_{i,n_i} + \tau) + c}} \frac{\Gamma[\eta(t_{i,n_i} + \tau) + c]}{\prod_{j=1}^{n_i+1} \Gamma(\Delta\eta_{i,j})} \exp(-sa) da}{\int_0^\infty \frac{a^{r-1} \prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j}-1}}{(w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i}) + c}} \frac{\Gamma[\eta(t_{i,n_i}) + c]}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})} \exp(-sa) da}, \quad (16)
\end{aligned}$$

where $t_{i,n_i+1} = t_{i,n_i} + \tau$, $w_{i,n_i+1} = w_{i,n_i} + \delta$, and $\Delta\eta_{i,n_i+1} = \eta(t_{i,n_i} + \tau) - \eta(t_{i,n_i})$. The degradation increment of the unit i over the future time interval now depends on the whole history of that unit up to t_{i,n_i} , but this does not mean that the physical degradation phenomenon is no longer Markovian. Indeed, the (not-physical) dependence of the degradation increment on the whole history arises from the fact that prediction relative to unit i is made by averaging over the subpopulation of units experiencing the same degradation history of unit i .

If only the scale parameter α is random, then the (conditional) distribution of α relative to the (heterogeneous) subpopulation of units whose vector of observed degradation is given by \mathbf{w}_i , is gamma:

$$f(\alpha | \mathbf{w}_i) = \frac{\alpha^{\eta(t_{i,n_i}) + c - 1} (w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i}) + c}}{\Gamma[\eta(t_{i,n_i}) + c]} \exp[-\alpha(w_{i,n_i}^\beta + d)], \quad (17)$$

with scale parameter $w_{i,n_i}^\beta + d$ and shape parameter $\eta(t_{i,n_i}) + c = a t_{i,n_i}^b + c$.

It is worthwhile to note that, in this case, the conditional distribution of α depends only on the last observation w_{i,n_i} , and not on the whole past history. The pdf of the degradation increment in the future time interval $(t_{i,n_i}, t_{i,n_i} + \tau)$ is then in closed form:

$$\begin{aligned}
f_{\Delta W}(t_{i,n_i}, \tau)(\delta | \mathbf{w}_i) &= \frac{\Gamma[\eta(t_{i,n_i} + \tau) + c]}{\Gamma(\Delta\eta_{i,n_i+1}) \Gamma[\eta(t_{i,n_i}) + c]} \frac{\beta w_{i,n_i+1}^{\beta-1} (w_{i,n_i+1}^\beta - w_{i,n_i}^\beta)^{\Delta\eta_{i,n_i+1}-1} (w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i}) + c}}{(w_{i,n_i+1}^\beta + d)^{\eta(t_{i,n_i} + \tau) + c}} \\
&= \frac{\beta w_{i,n_i+1}^{\beta-1} (w_{i,n_i+1}^\beta - w_{i,n_i}^\beta)^{\Delta\eta_{i,n_i+1}-1} (w_{i,n_i}^\beta + d)^{\eta(t_{i,n_i}) + c}}{B(\Delta\eta_{i,n_i+1}, \eta(t_{i,n_i}) + c) (w_{i,n_i+1}^\beta + d)^{\eta(t_{i,n_i} + \tau) + c}}, \quad (18)
\end{aligned}$$

where $B(\Delta\eta_{i,n_i+1}, \eta(t_{i,n_i}) + c)$ is the Beta function, and of course depends on the past history only through the last observation w_{i,n_i} .

Finally, if α is fixed and only a is random, $f(a | \mathbf{w}_i)$ depends on the whole history up to t_{i,n_i} , and the pdf of the degradation increment in the future time interval $(t_{i,n_i}, t_{i,n_i} + \tau)$ is given by:

$$\begin{aligned}
f_{\Delta W}(t_{i,n_i}, \tau)(\delta | \mathbf{w}_i) &= \int_0^\infty f_{\Delta W}(t_{i,n_i}, \tau)(\delta | w_{i,n_i}, a) f(a | \mathbf{w}_i) da \\
&= \beta w_{i,n_i+1}^{\beta-1} \exp[-\alpha(w_{i,n_i+1}^\beta + d)] \frac{\int_0^\infty \alpha^{\eta(t_{i,n_i} + \tau)} a^{r-1} \left(\frac{\prod_{j=1}^{n_i+1} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j}-1}}{\prod_{j=1}^{n_i+1} \Gamma(\Delta\eta_{i,j})} \right) \exp(-sa) da}{\int_0^\infty \alpha^{\eta(t_{i,n_i})} a^{r-1} \left(\frac{\prod_{j=1}^{n_i} [w_{i,j}^\beta - w_{i,j-1}^\beta]^{\Delta\eta_{i,j}-1}}{\prod_{j=1}^{n_i} \Gamma(\Delta\eta_{i,j})} \right) \exp(-sa) da} \quad (19)
\end{aligned}$$

4 Hypothesis testing

In order to test the hypothesis that one or both the scale parameters of the TG process are fixed, we note that the overdispersion (variance-to-mean) ratio of the gamma distributions (7) and (8) tends to 0 as $s \rightarrow 0$ or $d \rightarrow 0$, respectively, so that by letting, for example, $s \rightarrow 0$ with $r/s = E\{a\}$ fixed, the distribution (7) degenerates to a point mass at r/s , and hence the TG process with a random reverts back to the TG process with a fixed. Thus, the TG processes with a fixed, or with α fixed, or finally with a and α fixed, can be considered as asymptotically nested into the TG process with a and α random.

As a consequence, the likelihood ratio test can be used to test if a and/or α are fixed, against the alternative hypothesis that are random. For example, the likelihood ratio statistic

$$\Lambda = -2 \ln \frac{L(\mathbf{w} | \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{\beta})}{L(\mathbf{w} | \hat{r}, \hat{s}, \hat{b}, \hat{c}, \hat{d}, \hat{\beta})},$$

relative to the null hypothesis H_0 of a TG process with a fixed and α random, against the alternative hypothesis H_1 of a TG process with a and α random, is asymptotically distributed as a chi-squared random variables with $\nu=1$ degree of freedom, where $L(\mathbf{w} | \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{\beta})$ and $L(\mathbf{w} | \hat{r}, \hat{s}, \hat{b}, \hat{c}, \hat{d}, \hat{\beta})$ denote the estimated likelihood functions under the null and the alternative hypothesis, respectively.

In addition, because the TG process becomes gamma when $\beta=1$, then the likelihood ratio test can be easily used to assess if the observed process is gamma against the alternative hypothesis that the process is TG.

5 Numerical application

Let us consider the data set given in [7] that refers to the light intensity of 12 light emitting diodes (LED) that operate under a constant level of electric current of 40 mA. The degradation level of each unit, that is given by the percent loss of light intensity with respect to the initial intensity at time $t=0$, was measured every 50 hours until 250 hours (see Table 1).

The degradation paths are plotted in Figure 1, where the data points are linearly interpolated. The number of intersections of the plotted paths is very low, thus suggesting the possible existence of heterogeneity among the units.

Moreover, the empirical estimate of the variance of the degradation level, plotted in Figure 2, initially increases quickly, and then decreases monotonically with time. This behavior is not compatible with the gamma process, as well as with any other independent increment process whose variance increases monotonically with time, both in absence and in presence of heterogeneity. Thus, the TG process, whose variance is not constrained to increase monotonically with time, appears to be a suitable model for this data set.

Table 1: The LED degradation data

Time [h]	Units											
	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0
50	13.4	17.9	17.3	20.2	24.9	16.3	27.0	13.8	18.8	33.2	33.9	23.5
100	21.3	28.6	29.7	31.7	33.3	26.0	35.0	32.4	35.0	36.7	35.8	38.3
150	24.0	34.6	36.0	37.7	37.2	32.6	39.3	37.3	39.4	40.7	40.6	38.7
200	28.4	38.3	38.7	40.0	41.0	37.0	41.7	40.0	40.7	42.7	42.0	40.3
250	32.0	42.0	40.7	41.0	46.0	38.7	42.0	40.3	42.7	43.5	44.7	44.0

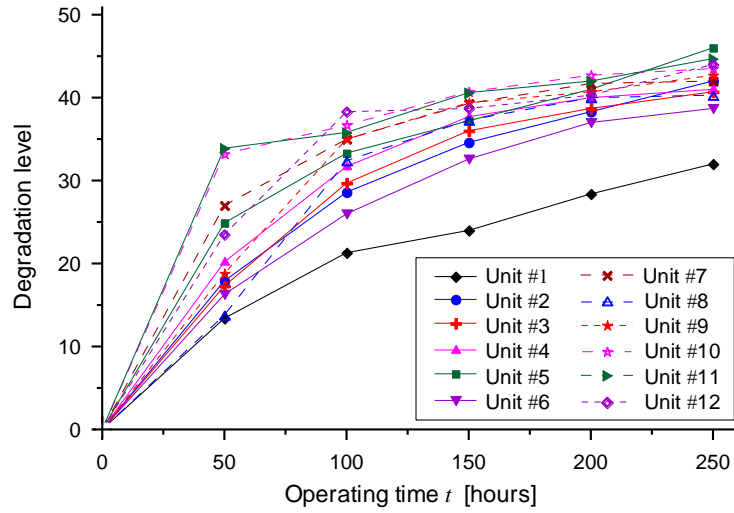


Figure 1. The observed degradation data.

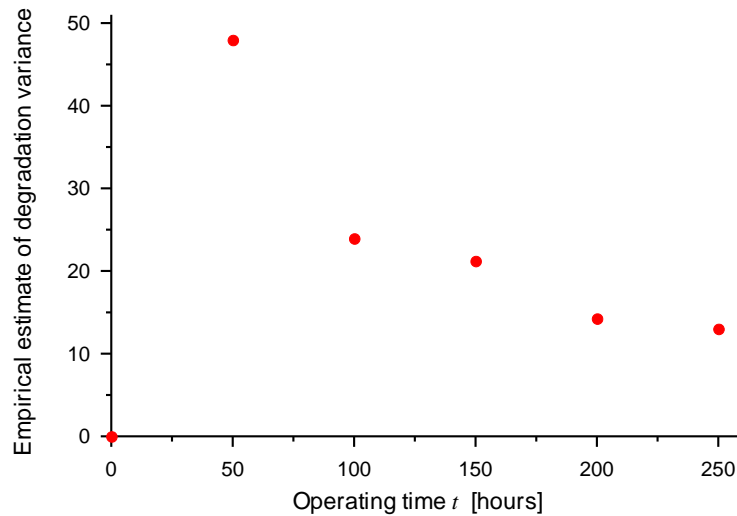


Figure 2. The empirical estimate of the degradation variance

Table 2 gives the maximum likelihood estimates of the TG parameters under several assumptions on the randomness of scale parameters a and α . Where a parameter is random, instead of giving the ML estimate of the hyperparameters of the corresponding gamma distribution, Table 2 gives the ML estimate of its mean, say $\hat{E}\{a\}=\hat{r}/\hat{s}$ or $\hat{E}\{\alpha\}=\hat{c}/\hat{d}$, and its coefficient of variation $\hat{\rho}\{a\}=1/\hat{r}^{1/2}$ or $\hat{\rho}\{\alpha\}=1/\hat{c}^{1/2}$. This choice makes easier the comparison with the cases where the parameter is fixed, and allows an immediate qualitative assessment of the randomness of the parameter a or α . The estimated log-likelihood functions and the values of the Akaike information criterion (AIC) are also provided. Note that, given a set of candidate models for the observed data, the preferred model is the one with the minimum AIC value [8].

From the AIC values in Table 2, we observe that the preferred TG model is Model #2, where only the scale parameter a of the age function $\eta(t)$ is random. In addition, we observe that the ML estimate of the coefficient of variation $\rho\{\alpha\}$ under Model #4, where a and α are both assumed to be random, is very small, say $\hat{\rho}\{\alpha\}=0.004$, thus confirming that the heterogeneity among the paths can be very well modeled by assuming that only a is random (Model #2).

Table 2: Estimation results

Model #	Random parameters	Maximum likelihood estimates						$\log \hat{L}$	AIC
		\hat{a} or $\hat{E}\{a\}$	$\hat{\rho}\{a\}$	\hat{b}	$\hat{\alpha}$ or $\hat{E}\{\alpha\}$	$\hat{\rho}\{\alpha\}$	$\hat{\beta}$		
1	None	$2.15 \cdot 10^{-2}$		1.08	$1.10 \cdot 10^{-6}$		4.24	-142.62	293.24
2	a	$8.11 \cdot 10^{-4}$	0.291	1.69	$2.51 \cdot 10^{-10}$		6.50	-139.30	288.61
3	α	$7.45 \cdot 10^{-4}$		1.68	$3.78 \cdot 10^{-10}$	0.252	6.37	-140.75	291.50
4	a & α	$8.15 \cdot 10^{-4}$	0.291	1.69	$2.56 \cdot 10^{-10}$	0.004	6.49	-139.30	290.61

Table 3: Likelihood ratio test results, at significance level of 0.10

Null hypothesis	Alternative hypothesis	Λ	ν	p -value	Test response
TG with a & α fixed	TG with a & α random	6.632	2	0.036	Reject H_0
TG with a random	TG with a & α random	0.000	1	1.000	Not reject H_0
TG with α random	TG with a & α random	2.892	1	0.089	Reject H_0
G with a & α random	TG with a & α random	22.730	1	$1.9 \cdot 10^{-6}$	Reject H_0
G with a random	TG with a random	22.715	1	$1.9 \cdot 10^{-6}$	Reject H_0

The same conclusion is reached by performing the likelihood ratio tests. Indeed, as shown in Table 3, at the significance level of 0.10, only the null hypothesis that the process is TG with a random and α fixed (Model #2) is not rejected against the TG alternatives. Also the null hypothesis that the process is Gamma (with a and α random, or with a random and α fixed) is rejected against the alternative hypothesis that the process is TG. This confirms what suggested by the empirical estimates of the degradation variance in Figure 2, that is, that the light intensity degradation process can not be adequately model by the gamma process.

Finally, in Figure 3 the ML estimates of the mean and variance of the observed process are plotted under Model #1 (TG process without random effects), under Model #2 (TG process with a random and α fixed), and under Model #4 (TG process with a and α random). Also the ML estimates under the gamma process with a and α random are depicted. We can see that the gamma process does not fit well the empirical mean and is totally unable to fit the empirical variance, whereas the TG process with fixed a and α (Model #1) is not able to fit well neither the empirical mean, nor the empirical variance. In addition the curves relative to the TG process with a and α random (Model #4) and the TG process with a random and α fixed (Model #2) overlap almost perfectly, and fit very well both the empirical mean and the empirical variance.

Finally, by using (19), the predictive pdf of the degradation increments $\Delta W(250,300)$ of selected units, namely units #1, 2, 3, 5, and 7, over the future time interval of width $\tau=50$ hours has been estimated and plotted in Figure 4. We note at first that the distribution of the degradation increments changes noticeable among the units, mainly due to the different degradation level reached by the units at the common time of the last inspection.

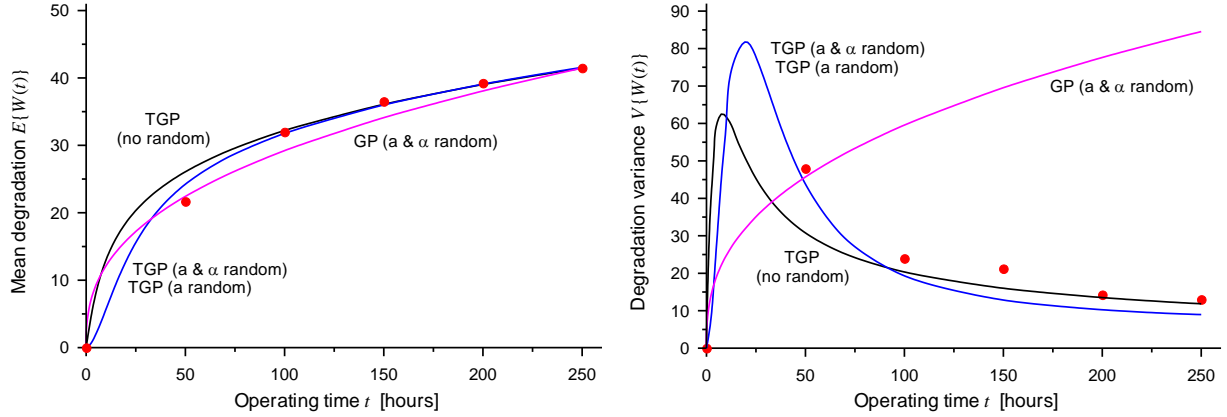


Figure 3: Comparison between empirical and ML estimate of the mean (on the left) and the variance (on the right) of the light intensity degradation process, under different models.

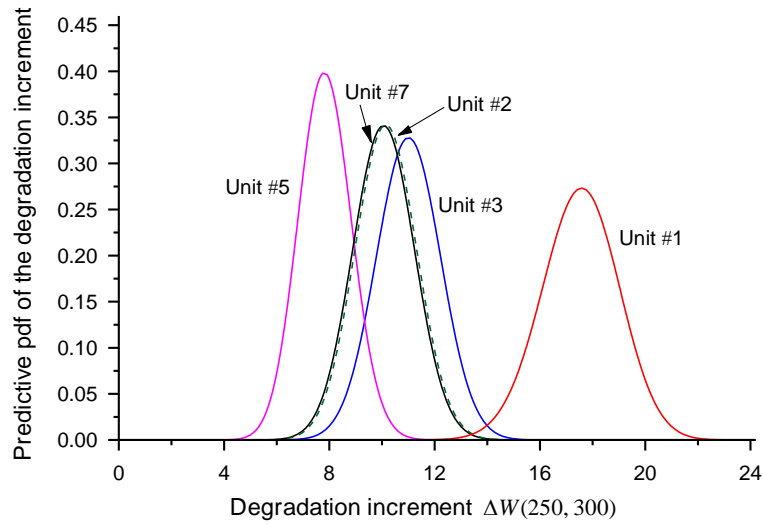


Figure 4: ML estimate of the predictive distribution of the degradation increments of selected units over the future interval of 50 h.

Moreover, we note that a small difference exists also between units #2 and 7, although their degradation level at the last inspection time is the same. This is due to the fact that the past histories of these units are different and, although the process is physically assumed to be Markovian, the distribution of the future degradation increment depends on the whole history of the unit, due to the dependence of $g(a | \mathbf{w}_i)$ on the whole history. The (small) difference in the predictive distribution implies a slight difference in the estimate of the mean increment, say $\hat{E}\{\Delta W(250, 300) | \mathbf{w}_2\} = 10.15$ and $\hat{E}\{\Delta W(250, 300) | \mathbf{w}_7\} = 10.06$.

6 Conclusions

In this paper the presence of random heterogeneity among units subject to a degradation process with dependent increments has been considered. The degradation process has been described by the recently proposed transformed gamma process, a Markovian process that allows dependence between degradation increments over disjoint time intervals to be easily modeled. The random heterogeneity, that can arise from random operating

conditions and/or technological differences among the units, has been modeled by assuming that the scale parameters of the two functions which index the stochastic process model, namely the age function and the state function, are independent, gamma distributed random variables. Some distributional characteristics of the proposed random-effects model have been discussed, in particular the conditional distribution of the degradation increment over a future time interval of a given unit that, in some cases, can depend on the whole past history of that unit, although the degradation process is physically Markovian.

The application of the proposed random-effects model to a real data set has shown the feasibility of the model and its ability to assess which model parameter is really affected by heterogeneity.

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