Bayesian degradation testing planning for Wiener process

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This paper describes Bayesian method for degradation test planning with Wiener process. We useconjugate prior distributions and criteria based on estimating a unilateral confidence interval of reliability metric. One criterionis based on a precision for a credibility interval for failure probability. We provide simple closed form expressions for the relationshipbetween the needed numbers of paths and measures(total number of degradation increments) and the precision criteria. An example is usedto illustrate the method.

*Keywords:*Bayes, Testing planning, Degradation, Wiener Process, Reliability

1 Introduction

Reliability assessment is becoming an integral part of the design process of complex systems in order to highlight potential risk areas so they can be dealt with at the design stage of the project. Indeed, the early control of system specifications allows diminishing operating (either financial or safety) risks. Since systems must be more and more reliable and offer longer guarantees, it is necessary to check the compliance of their performances as early as possible.

One can analyze two failure types:

- Material failures, often appearing all of a sudden,
- Soft failures, meaning a performance drift in time, until unacceptable levels.

Testing prototypes allows evaluating the reliability of a system before it is mass-produced. This process requires long testing times and huge numbers of prototypes since the latter are more and more reliable, therefore extremely diminishing the probability of failure.
One alternative solution is the study of a performance drift in time, in order to characterize failure probability. This is done by testing a number of systems and by measuring the evolution of their performance in time, z(t). The systems are considered as failed when their performance has reached the critical value denoted z_c.

The constantly increasing market request for high quality device to verify, before starting mass production, if new components or parts attain a field reliability target. To this end, reliability testing is used to estimate the lifetime distribution [1]. Common problems in lifetime distribution estimation by testing are the total time required to test and the available number of examples for testing to demonstrate reliability to a customer’s satisfaction. This paper proposes to define a degradation test plan based on Wiener process and Bayes estimation [1, 3].

2 Reliability testing

Degradation tests for reliability estimation consist in measuring the evolution of the degradation during the testing of a sample of products or systems. We thus obtain a degradation path, z(t), for each tested system and a network of degradation paths for the entire sample (see Figure1).

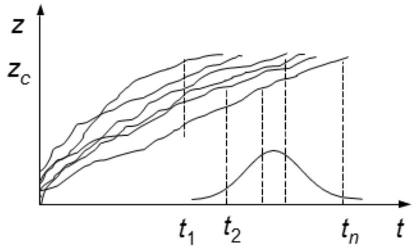


Figure 1- Degradation paths example

The system is considered as failed when its degradation reaches a critical value, denoted z_c. Reaching this critical value allows obtaining pseudo-failure times, denoted t_i, which are then used to assess reliability function.

Degradation processes are paths of some stochastic process with independent increments. Wiener process [4, 8] characterizes average monotonic degradations. In this paper, we consider the case of a Wiener process with linear leaning μ and variance σ^2, with following hypothesis:

- W(0)=0,
- Increment law $W(t+h)-W(t)$ is normal distribution $N(\mu h, \sigma^2 h)$,
- If W_0 is a standard Wiener process, i.e. $\mu = 0$ et $\sigma^2 = 1$, then $W(t) = \mu t+\sigma W_0(t)$ is a Wiener process of linear leaning μ and variance σ^2.

The distribution of pseudo instants of failure, T, is an inverse normal distribution IG(z_c/μ, z_c^2/σ^2), of density given by:

$$f(T\,|\,z_c,\mu,\sigma)=\frac{z_c}{\sqrt{2\pi\sigma}}T^{-\frac{3}{2}}e^{\left(-\frac{(z_c-\mu t)^2}{2\sigma^2 T}\right)} \qquad (1.)$$

The estimation of $\mu$and σ is obtained by maximum likelihood, using the observed increments; the degradation increments are denoted Δz_{ij} (for path *i* (*m* paths) and time *j* ($q_i$measures on path), as shown Figure2).

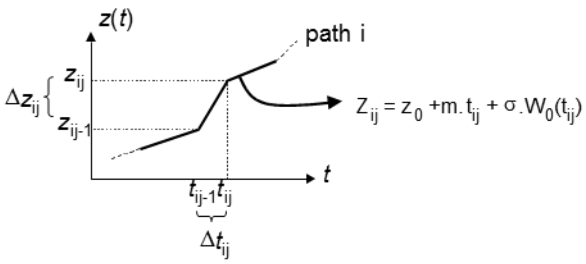


Figure 2 - Example of data

Since degradation increment Δz_{ij} is characterized by a normal distribution (of mean $\mu.\Delta t_{ij}$ and variance $\Delta t_{ij}.(1/\theta^2)$ with $\sigma^2=1/\theta^2$), the likelihood is:

$$g(\Delta z\,|\,\mu,\theta)=\prod_{i=1}^{m}\prod_{j=1}^{qi}\frac{\theta^{\frac{1}{2}}}{\sqrt{2\pi\Delta t_{ij}}}e^{-\left(\frac{\theta\left(\Delta z_{ij}-\mu\Delta t_{ij}\right)^2}{2\Delta t_{ij}}\right)} \qquad (2.)$$

In test, the periodicity of degradation measurements is often constant ($\Delta t_{ij} = \Delta t$). Considering this assumption, the likelihood function can be written

$$g(\{x\}\,|\,\mu,\theta)=\prod_{i=1}^{m}\prod_{j=1}^{qi}\frac{\theta^{\frac{1}{2}}}{\sqrt{2\pi}}e^{-\left(\frac{\theta\left(x_{ij}-\mu\right)^2}{2}\right)} \qquad (3.)$$

with$x_{ij} = \Delta z_{ij}/\Delta t$

The function $g(\{x\}|\mu, \theta)$ is characteristic of normal likelihood function.Usually, a probability distribution is defined by its parameters which are often unknown constants. Based on a random sample, one can use the maximum likelihood method to estimate and obtain confidence intervals for the parameters and the reliability function.

From a random sample of n ($n=\sum_{i=1}^{m} q_i$) observations the sample mean $\bar{x}$ and sample standard deviation s are computed (point estimates) [3, 5, 6]:

$$\bar{x}=\frac{\sum_{i=1}^{m}\sum_{j=1}^{q_i} x_{ij}}{n} \text{ and } s=\sqrt{\frac{\sum_{i=1}^{m}\sum_{j=1}^{q_i}\left(x_{ij}-\bar{x}\right)^2}{n-1}} \qquad (4.)$$

3 Bayesian estimation

Recently, a rising interest in the Bayesian approach to reliability and life parameter estimation has emerged [2]. To statisticians and reliability engineers this approach is appealing since it provides a method of using their past experiences and/or prior convictions in describing the studied parameter x stochastically.
On some situations the parameter is not known, but can be treated as a random variable with a known prior probability density. Under this scenario, one can combine information from the random sample and prior probability distributions to obtain (l-γ) Bayesian confidence intervals for the parameters. The objective of this section is to obtain the Bayesian estimators for the parameters μ and θ of the normal distribution [5, 6].

3.1 Bayesian principle

The probability density function $f(\mu,\theta\,|\{x\})$ of the posterior *pdf* of μ and θ obtained from the sample of observations {x} and the *pdf* $f(\mu)$ and $f'(\theta)$ of the prior distribution of μ and θ is given by

$$f(\mu,\theta\,|\,\{x\})=\frac{g(\{x\}\,|\,\mu,\theta)\cdot f(\mu)\cdot f'(\theta)}{\int_{D(\mu)}\int_{D(\theta)} g(\{x\}\,|\,\mu,\theta)\cdot f(\mu)\cdot f'(\theta)\cdot d\mu d\theta} \qquad (5.)$$

where μ and θ: parameters to estimate

$\{x\}=\{x_{11}, \ldots, x_{1q_1}, \ldots, x_{ij}, \ldots, x_{mq_m}\}$: observed data

$f(\mu)$ and $f'(\theta)$: prior probability density functions (available knowledge from the experts)

$g(\{x\}|\mu,\theta)$: likelihood function

$f(\mu,\theta\,|\{x\})$: posterior density function

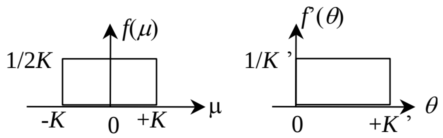
$D(\mu)$ and $D(\theta)$: set of nature states

Now, two cases are studied to define the posterior *pdf:*

- No knowledge on μ and θ
- Available knowledge on μ and θ

Case 1: No knowledge on μ and θ

When there is no information about the mean μ and the inverse variance θ, the uniform uninformative density is used to define the prior *pdf* $f(\mu)$ and $f'(\theta)$. The selection of this uniform probability density is based on the fact that this *pdf* has maximum entropy among all *pdf* that are non zero in a given range.



with $K, K'\to\infty$

Figure3 - Uniform uninformative pdf

The likelihood function is given for a sample of size n by

$$g\left(\{x\}\,|\,\mu,\theta\right)=\frac{\theta^{\frac{n}{2}}}{\prod_{i=1}^{n}\prod_{j=1}^{q_i}\sqrt{2\pi}}\, e^{-\frac{1}{2\Delta t_{ij}}\theta\sum_{i=1}^{n}\left(\Delta x_{ij}-\mu\right)^2} \qquad (6.)$$

Thus, the posterior *pdf* is written as

$$f(\mu,\theta\,|\,\{x\})=\frac{g\left(\{x\}\,|\,\mu,\theta\right) f(\mu) f(\theta)}{\int_{-K}^{+K}\int_{0}^{+K'} g\left(\{x\}\,|\,\mu,\theta\right) f(\mu) f(\theta) d\mu d\theta} \qquad (7.)$$

$$\text{with } K, K'\to\infty$$

Following [3, 5, 6], the posterior *pdf* is given by

$$f\left(\mu,\theta\,|\,\{x\}\right)=\underbrace{\frac{b^{a}}{\Gamma(a)}\theta^{a-1}e^{-b\theta}}_{\mathcal{G}(a,b)}\underbrace{\sqrt{\frac{n\theta}{2\pi}}e^{-\frac{n\theta}{2}(\mu-c)^{2}}}_{\mathcal{N}\left(c,\frac{1}{\sqrt{\theta(2a-1)}}\right)} \qquad (8.)$$

where

$$a=\frac{n+1}{2} \qquad b=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{q_i}\left(x_{ij}-\bar{x}\right)^{2} \qquad c=\bar{x}$$

which is a combination of the gamma and normal distributions.

Case 2: Available knowledge about μ and θ

Following [3, 5, 6], we propose to choose the prior *pdf* defined by the relationship (8.). Then the posterior *pdf* is written(9.)

$$f\left(\mu,\theta\,|\,\{x\}\right)=\underbrace{\frac{b'^{a'}}{\Gamma(a')}\theta^{a'-1}e^{-b'\theta}}_{\mathcal{G}(a',b')}\underbrace{\sqrt{\frac{(2a'-1)\theta}{2\pi}}e^{-\frac{(2a'-1)\theta}{2}(\mu-c')^{2}}}_{\mathcal{N}\left(c',\frac{1}{\sqrt{\theta(2a'-1)}}\right)} \qquad (9.)$$

where

$$a'=\frac{n}{2}+a \qquad c'=M=\frac{n\bar{x}+(2a-1)c}{n+2a-1} \qquad b'=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{q_i}\left(x_{ij}-\bar{x}\right)^{2}+b+\frac{n(2a-1)(\bar{x}-c)^{2}}{2(n+2a-1)}$$

which is a combination of the gamma and normal distributions.

Note that the form of relationship (9.) is identical to (8.). The prior *pdf* defined by (9.) is the natural conjugate.

3.2 Bayesian point and interval estimation

Once the posterior distribution is defined, the estimators of the parameters μ and θ can be obtained by using the marginal distributions associated to μ and θ.

The marginal distribution of θ is written as

$$f\left(\theta\,|\,\{x\}\right)=\int_{-\infty}^{+\infty} f(\mu,\theta\,|\,\{x\})d\mu$$

$$f\left(\theta\,|\,\{x\}\right)=\mathcal{G}\left(a',b'\right)\int_{-\infty}^{+\infty}\mathcal{N}\left(c',\frac{1}{\theta(2a'-1)}\right)d\mu \qquad (10.)$$

$$f\left(\theta\,|\,\{x\}\right)=\frac{b'^{a'}\theta^{a'-1}e^{-b'\theta}}{\Gamma(a')}=\mathcal{G}(a',b')$$

The point estimate for θ is defined by the mode of $f(\theta|x)$ and the point estimate of variance are given by

$$\hat{\theta}=\frac{a'-1}{b'} \text{ and } \hat{s}^{2}=\frac{b'}{a'-1} \qquad (11.)$$

The two-sided confidence interval (defined by θ_{min} and θ_{max}) is evaluated such that:

$$\begin{cases}\dfrac{\gamma}{2}=\displaystyle\int_{0}^{\theta_{\min}} f(\theta\,|\,\{x\})d\theta & \Rightarrow \quad \theta_{\min}\\[2ex] 1-\dfrac{\gamma}{2}=\displaystyle\int_{0}^{\theta_{\max}} f(\theta\,|\,\{x\})d\theta & \Rightarrow \quad \theta_{\max}\end{cases} \qquad (12.)$$

with (1-γ) the given confidence level.

The marginal distribution of μ is written

$$f\left(\mu\,|\,\{x\}\right)=\int_{0}^{+\infty} f(\mu,\theta\,|\,\{x\})d\theta=\frac{b^{a}\sqrt{2a-1}}{\alpha^{a+\frac{1}{2}}\sqrt{2\pi}}\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)} \qquad (13.)$$

with $\alpha=b+\frac{1}{2}(2a-1)(\mu-c)^{2}$

The point estimate for μ is defined by the mode of $f(\mu|\{x\})$

$$\hat{\mu}=c' \qquad (14.)$$

The two-sided confidence interval (defined by μ_{min} and μ_{max}) is evaluated such that

$$\begin{cases} \frac{\gamma}{2} = \int_0^{\mu_{\min}} f(\mu \mid \{x\}) d\mu & \Rightarrow \quad \mu_{\min} \\ 1-\frac{\gamma}{2} = \int_0^{\mu_{\max}} f(\mu \mid \{x\}) d\mu & \Rightarrow \quad \mu_{\max} \end{cases} \quad (15.)$$

with (1-γ) the given confidence level.

In the case when no knowledge is available, the point estimates of mean and variance are defined by known relationships

$$\hat{\mu} = c = \bar{x} \text{ and } \hat{s}^2 = \frac{b}{a-1} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(x_{ij} - \bar{x}\right)^2}{n-1} \quad (16.)$$

with$a$, b and c defined by eq (9.)

In the case in which prior knowledge is available, the point estimates of mean and variance are defined by :

$$\hat{\mu} = c' = \frac{n\bar{x} + (2a-1)c}{n + 2a - 1} \quad (17.)$$

and

$$\hat{s}^2 = \frac{b'}{a'-1} = \frac{\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{q_i} \left(x_{ij} - \bar{x}\right)^2 + b + \frac{n(2a-1)(\bar{x}-c)^2}{2(n+2a-1)}}{\frac{n}{2} + a - 1} \quad (18.)$$

where a, *b* and *c* are the parameters of the prior *pdf* and *a*', *b*' and *c*' the parameters of the posterior *pdf.*

3.3 Determination of the prior distribution from available information

The standard deviation (*s*) interval and prior mean degradation increment μ are provided by an expert [7] or the results of a previous analysis. The prior knowledge is given by a believed estimation of mean and a range believed to contain the inverse of the variance

$$[\mu] \text{ and } \left[\theta_{\min} = \frac{1}{s^2_{\max}}, \theta_{\max} = \frac{1}{s^2_{\min}}\right]$$

The prior *pdff(μ,θ/x)* is defined by the relationship (10.) with the unknown parameters *a, b* and *c*.

Evaluation of a and b by Moments method

The marginal distribution of θ is written

$$f\left(\theta \mid \{x\}\right) = \frac{b^a \theta^{a-1} e^{-b\theta}}{\Gamma(a)} = \mathcal{G}(a,b) \quad (19.)$$

The interval [$\theta_{\min}$, $\theta_{\max}$] defines an uniform distribution. The mean and variance for this distribution are

$$E(\theta) = \frac{\left(\theta_{\min} + \theta_{\max}\right)}{2} \text{ and } V(\theta) = \frac{\left(\theta_{\max} - \theta_{\min}\right)^2}{12} \quad (20.)$$

The mean and variance for a gamma distribution $\mathcal{G}$(a,b) are given by

$$E'(\theta) = \frac{a}{b} \text{ and } V'(\theta) = \frac{a}{b^2} \quad (21.)$$

By evaluating of means and variances ($E(\theta) = E'(\theta)$ and $V(\theta) = V'(\theta)$), the values of parameters a and b are deducted to be

$$a = \frac{3\left(\theta_{\min} + \theta_{\max}\right)^2}{\left(\theta_{\max} - \theta_{\min}\right)^2} \text{ and } b = \frac{6\left(\theta_{\min} + \theta_{\max}\right)}{\left(\theta_{\max} - \theta_{\min}\right)^2} \quad (22.)$$

Evaluation of c

The marginal distribution of μ is written

$$f\left(\mu \mid \{x\}\right)=\frac{b^{a}\sqrt{2a-1}}{\alpha^{a+\frac{1}{2}}\sqrt{2\pi}}\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)} \qquad (23.)$$

with $\alpha = b+\frac{1}{2}(2a-1)(\mu-c)^2$.

This marginal distribution is symmetric around the c value (a and b are also defined). Then the c value is given by

$$c=\mu \qquad (24.)$$

In the following section, we propose to solve the degradation test planning problem for Wiener process.

4 Test planning

Degradation tests are often planned to estimate a particular metric (i.e.,failure probability P_f) of the reliability distribution (see figure 4). It is natural to use a criterion for the planning problem that is constructed from somemeasure of the precision of estimation of P_f.

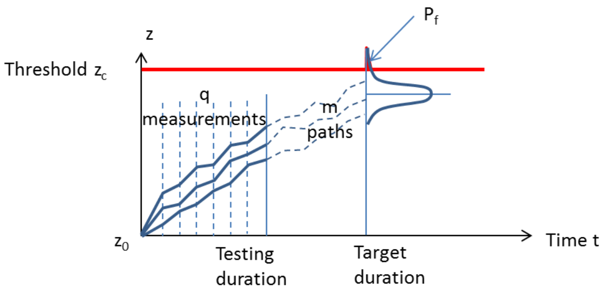


Figure 4 – Definition of planning problem

In the Bayesian framework, estimation precision can be specified as a functionof the posterior estimation of reliability metric. For a given test planD(m, q_i with i=1,…m, testing duration T), the posterior estimation of reliability metric depends on the data. A reasonable Bayesian criterion for test planning is then the preposterior expectationof the posterior estimation precision function. This criterion is computed by taking an expectation over the marginal distribution of the data to account for allpossible outcomes from theexperiment.

The failure probability is given by :

$$P_f = 1-\Phi(u) \qquad (25.)$$

where: $u=\frac{z_c-[\hat{\mu}*t+z_0]}{s*\sqrt{t}}$ with target time t , initial degradation $z_0$and critical degradation thresholdz_c

The upper interval on failure probability (for confidence interval 1-α)is :

$$Pf_{upper} = 1-\Phi(u_{lower}) \qquad (26.)$$

with:

$$u_{lower} = u - k_{1-\alpha} * \sqrt{var(u)}$$

and

$$Var(u) = \frac{1}{s^2} * [\sqrt{t} * Var(\mu) + u^2 * var(s)]$$

with$k_{1-\alpha}$value of the quantile of normal standard distribution for the probability 1-α.

The variance on u is deducted from (10.) and (13.) :

$$Var(u)=\frac{1}{b'}\left[\sqrt{\frac{b't}{(2a'-1)}}+\frac{a'\left[z_c-\left(c't+z_0\right)\right]^2}{4(a'-1)}\right] \qquad (27.)$$

where

$$a'=\frac{n}{2}+a \qquad c'=M=\frac{n\bar{x}+(2a-1)c}{n+2a-1} \qquad b'=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{q_i}\left(x_{ij}-\bar{x}\right)^2+b+\frac{n(2a-1)(\bar{x}-c)^2}{2(n+2a-1)}$$

To define the test plan, we determine the lower bound u_{lower}on u from failure probability target P_f:

$$u_{lower} = \Phi^{-1}(1 - Pf) \qquad (28.)$$

The target variance $\tilde{V}ar(u)$ on u is deducted from :

$$\tilde{V}ar(u) = \sqrt{\frac{u - u_{lower}}{k_{1-\alpha}}} \qquad (29.)$$

Where $u = \frac{z_c - [\hat{\mu} * t + z_0]}{s * \sqrt{t}}$ is defined with the mean μ and standard deviation $s$given, respectively, by the relationships (17.) and (18.).

We consider that the testing results will follow the prior degradationincrement distribution :

$$\bar{x} = c \quad and \quad \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{q_i}\left(x_{ij} - \bar{x}\right)^2 = b \qquad (30.)$$

Finally, the posterior parameters become :

$$a' = \frac{n}{2} + a \qquad c' = \frac{(n + 2a - 1)c}{n + 2a - 1} \qquad b' = 2b \qquad (31.)$$

The testing plan is chosen in searching the value n in relationship (27.) allowing to guarantee the value $\tilde{V}ar(u)$ (see eq. 29.).

5 Example

Let us consider Negative Temperature Coefficient (NTC) probes (see Figure 5).

Figure 5 -Negative Temperature Coefficient (NTC) probes and test conditions

A classic test is to put it in a climatic chamber and wait for humidity infiltration that will lead to a drift on resistance and so finally on temperature value (see Figure 5).

From literature and experience, NTC references that have troubles are chosen with:

- Mean μ = 0.37 K/day
- Standard deviation σ = 0.2 K/day$^{1/2}$
- Critical value z_c is chosen at 2 K.

The target failure probability P_f is fixed at 10% for duration t=4 days and risk α = 20%.

The prior parameters are determined in considering a variability of 30% on inverse of variance.

μ	**θ_{min}**	**θ_{max}**
0.37 K/day	0	50
a (from eq. (22.))	**b** (from eq. (22.))	**c** (from eq. (24.))
3	0.24	0.37

Table 1–Prior information and parameters

The lower bound u_{lower} on u is determined from failure probability target $P_f$by eq. (28.) :

$$u_{lower} = \Phi^{-1}(1 - 0.1) = 1.28$$

The target variance $\tilde{V}ar(u)$ on u is deducted from eq. (29.) :

$$\tilde{V}ar(u) = \sqrt{\frac{u - u_{lower}}{k_{1-\alpha}}} = \sqrt{\frac{1.3 - 1.28}{0.84}} = 0.148$$

The testing plan is chosen in searching the value n in relationship (27.) allowing to guarantee the previous value $\tilde{V}ar(u)$. The result is given in table 2.

a'	**b'**	**c'**	**n**	**Var(u) (from eq. (27.)**
27.5	0.24	0.37	49	0.147 (<$\tilde{V}ar(u)$ =1.48)

Table 2–posterior parameters and sample size of degradation increments

In considering industrial constraints, the items sample size and measures number can be chosen (ie. 5 items and 10 measures per item).

6 Conclusion

In this paper, Bayesian method for degradation test planning with Wiener process is presented. We useconjugate prior distributions and criteria based a credibility interval for failure probability. We provide simple closed form expressions for the relationshipbetween the needed numbers of paths and measures(total number of degradation increments) and the precision criteria. An example is usedto illustrate the method.

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