Sensitivity analysis for the block replacement policy

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This paper proposes a sensitivity analysis of the block replacement policy when the unknown parameters of the time-to-failure are unknown but can be estimated. Based on the asymptotic normality of the parameters estimator and the δ-method, the asymptotic distribution of various quantities is derived and therefore the sensitivity of the block replacement policy to the parameter estimations is analyzed. An application to the case of exponentially distributed lifetime is proposed.

Keywords: Asymptotic normality; δ-method; Exponential distribution; Implicit equation

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1 Introduction

For a given parametric distribution of time-to-failure (or lifetime) of a device, one can determine the optimal inter-inspection delay for the block replacement policy. However, this optimal block replacement policy depends on the parameters of the time-to-failure distribution, which are in general unknown. These parameters can be estimated, for instance, from a sample of time-to-failure observations and it provides only an estimation of the optimal inter-inspection delay (still for the block replacement policy). A question arises naturally: what is the variability induced by the estimation procedure? The sensitivity analysis of the block replacement policy is the purpose of this paper.

In Section 2, we start to recall some well-known results for the block replacement policy. Then, we provide some general results about the sensitivity analysis of this replacement policy. These results are based on the assumption of asymptotic normality of the estimators. It allows to apply the classical δ-method and some of its extensions. In Section 3, we provide an application to the case of the exponential distribution for the time-to-failure. In the last section, we conclude with some forthcoming works.

2 Sensitivity analysis of the block replacement policy

In this section, we first recall some definitions and well-known results corresponding to the block replacement policy. Then, we consider its sensitivity analysis. It will be based on the application of a generalization of the δ-method. Eventually, we provide some general results.

3 Recall on block replacement policy

Well-known results on block replacement policy are provided here. For more details, one can refer to chapter 5 in [3].

Let T be the time-to-failure (or lifetime) of the device. Suppose that the distribution of T depends on some parameter $\theta \in \Theta \subset \mathbb{R}^p$. One considers the usual notation: $f_T(\cdot;\theta)$ for its probability distribution function, $F_T(\cdot;\theta)$ for its cumulative distribution function, and $S_T(\cdot;\theta)$ for its survival function (or reliability).

One assumes that the degradation level of the device can be measured only during inspections (i.e. no continuous monitoring) and that, at each replacement, the device is replaced by a new one or is perfectly repaired (AGAN), the replacement/repair duration being negligible. Moreover, replacement occurs only after an inspection (in particular there is no replacement at times-to-failure). Such scheme is the so-called block replacement policy. There exists two different costs : the cost c_r for replacing the device by a new one and the unavailability cost c_u.

This policy depends on a single parameter δ, the delay between two consecutive inspections and induces a certain cost. Thus, one can be interested in determining the optimal inter-inspection delay $\delta^\star$. In order to determine it, we consider the asymptotic cost per unit of time defined as follows:

$$C(\delta;\theta) = \lim_{t\to\infty} \frac{C_t(\delta)}{t},$$

where $C_t(\delta)$ is the cost over the time interval $[0,t]$ when the device is inspected at $(k\delta)_{k\in\mathbb{N}}$. As it is well known [2], according to the renewal theory, one has:

$$C(\delta;\theta) = \frac{\text{Expected cost over a cycle}}{\text{Expected cycle length}}.$$

For the block-replacement policy, we have:

$$C(\delta;\theta) = \frac{\mathbb{E}[c_r + c_u(\delta - T)_+]}{\delta} = \frac{c_r + c_u \int_0^\delta F_T(u;\theta)\mathrm{d}u}{\delta}.$$

Therefore,

$$\lim_{\delta\to 0} C(\delta;\theta) = +\infty \quad \text{and} \quad \lim_{\delta\to+\infty} C(\delta;\theta) = c_u.$$

Let us denote $\delta^\star := \text{argmin}_{\delta>0} C(\delta;\theta)$. Differentiating the above expression of the cost function, $\delta^\star$ is the root of the following function (with respect to δ):

$$\phi(\delta;\theta) = \mathbb{E}[T\mathbf{1}_{T\leqslant\delta}] - \frac{c_r}{c_u} \tag{1}$$

where

$$\mathbb{E}[T\mathbf{1}_{T\leqslant\delta}] = \int_0^\delta u f_T(u;\theta)\mathrm{d}u = -\delta S_T(\delta;\theta) + \int_0^\delta S_T(u;\theta)\mathrm{d}u \tag{2}$$

(recalling that $\frac{c_r}{c_u} \leqslant 1$). The solution depends on the two costs c_r and c_u only through their ratio (and so does not depend on the monetary unit). Since, for fixed $\theta\in\Theta$, $\delta\mapsto\phi(\delta;\theta)$ is an increasing function towards $\mathbb{E}[T] - c_r/c_u$ (which depends on θ) with $\phi(0;\theta) = -c_r/c_u$, it follows that $\delta^\star$ is finite if $\mathbb{E}[T] > c_r/c_u$ and is infinite otherwise. Under this condition of existence for $\delta^\star$, it is well-known that the optimal cost is equal to:

$$C^\star = C(\delta^\star;\theta) = c_u F_T(\delta^\star;\theta).$$

It follows that the optimal delay is given by:

$$\delta^\star = F_T^{-1}(C^\star/c_u;\theta),$$

where $F_T^{-1}(\cdot;\theta)$ is the quantile function of the random variable T. Replacing this expression of the optimal delay in the function ϕ and after some simple algebra, one can obtain an implicit function ψ satisfied by $C^\star$:

$$\psi(C^\star;\theta) = \int_0^{C^\star/c_u} F_T^{-1}(u;\theta)\mathrm{d}u - \frac{c_r}{c_u} = 0.$$

3.1 General results for the sensitive analysis

Let us denote by $\theta_0 \in \Theta$ the true parameter of the time-to-failure distribution. As seen above, optimal inter-inspection delay $\delta_0^\star$, can be computed by determining the root of $\phi(\cdot;\theta_0)$. Most of the time, the parameter θ_0 is unknown but could be estimated from a sample of time-to-failure observations. Let us denote by $\widehat{\theta}_n$ an estimator of θ, say for instance the maximum likelihood estimator. Assume that this estimator has 'good' properties, like consistency:

$$\widehat{\theta}_n \xrightarrow[n\to\infty]{Pr} \theta_0, \tag{3}$$

and like asymptotic normality:

$$\sqrt{n}\left(\widehat{\theta}_n - \theta_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \Sigma_0^2), \tag{4}$$

where the asymptotic variance-covariance matrix Σ_0^2 depends on θ_0. If θ_0 is unknown, one can replace θ_0 by $\widehat{\theta}_n$ in order to estimate $\delta_0^\star$. In such case, we have only an estimator $\widehat{\delta}_n^\star$ of the optimal inter-inspection delay. A natural problem is then the following: which properties are satisfied by $\widehat{\delta}_n^\star$? The optimal cost is unknown but could be estimated by $\widehat{C}_n^\star$: which properties of $\widehat{C}_n^\star$ also hold? How far is this estimation from $C_0^\star$?

A convenient tool to answer these questions will be the δ-method. It can be stated as follows:

Theorem 3.1 *Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{R}^p$-valued random vectors. Assume there exists $\mu_X \in \mathbb{R}^p$ and Σ a definite positive matrix such that*

$$\sqrt{n}\,(X_n - \mu_X) \xrightarrow[n\to\infty]{d} N(0,\Sigma).$$

Let q real functions $f_1,\ldots,f_q$ with continuous first partial derivatives at μ_X, where at least one of these derivatives is non-zero. For $i \in \{1,\ldots,q\}$ and for any $n \in \mathbb{N}^*$, set $Y_{i,n} = f_i(X_n)$, $Y_n = (Y_{1,n},\ldots,Y_{q,n})^T$ and $\mu_X = (f_1(\mu_X),\ldots,f_q(\mu_X))^T$. Then, the sequence $(Y_n)_{n\in\mathbb{N}^*}$ is also asymptotically normal:

$$\sqrt{n}\,(Y_n - \mu_Y) \xrightarrow[n\to\infty]{d} N(0, K\Sigma K^T),$$

where K is the $q\times p$ matrix with elements $k_{i,j} = \partial f_i/\partial x_j$ for $i \in \{1,\ldots,p\}$ and $j \in \{1,\ldots,q\}$.

This result will be used here to obtain a point-wise asymptotic normality of the cost function (i.e. for any fixed value of δ). Unfortunately, we cannot apply it to derive the asymptotic normality for the optimal delay or for the optimal cost. Indeed, $\delta_0^\star$ and $C_0^\star$ are the solutions of implicit equations and the classical δ-method cannot be anymore used. Thus, we need an analogous tool for this situation. It has been considered by Benichou and Gail [1].

Theorem 3.2 *Let $(X_n)_{n\in\mathbb{N}^*}$ be as in the previous theorem. Let $\mu_X \in \mathbb{R}^p$ and $\mu_Y \in \mathbb{R}^q$. Let $g_1,\ldots,g_q$ a set of q continuous functions from $\mathbb{R}^p \times \mathbb{R}^q$ into $\mathbb{R}$ with continuous first partial derivatives in an open set containing (μ_X,μ_Y). Let Y_n be the $\mathbb{R}^q$-valued random vectors satisfying $g_r(X_n,Y_n) = 0$ for all $r \in \{1,\ldots,q\}$. Let $J_{x,y}$ be the $q\times q$ matrix with elements $\frac{\partial g_i}{\partial y_j}(x,y)$ and let $H_{x,y}$ be the $q\times p$ matrix with elements $\frac{\partial g_i}{\partial x_j}(x,y)$. If $|J_{\mu_X,\mu_Y}| \neq 0$ and if each rows of $J^{-1}_{\mu_X,\mu_Y}H_{\mu_X,\mu_Y}$ contain at least one nonzero element, then*

$$\sqrt{n}\,(Y_n - \mu_Y) \xrightarrow[n\to\infty]{d} N(0, J^{-1}_{\mu_X,\mu_Y}H_{\mu_X,\mu_Y}\Sigma H^T_{\mu_X,\mu_Y}(J^{-1}_{\mu_X,\mu_Y})^T).$$

We first consider the estimated cost function obtained by the plug-in method:

$$\forall\delta > 0, \quad C(\delta;\widehat{\theta}_n) = \frac{c_r + c_u\int_0^\delta F_T(u;\widehat{\theta}_n)\mathrm{d}u}{\delta}.$$

By a simple application of the continuous mapping theorem and provided that Equation (3) is satisfied, it is a convergent point-wise estimator of $C(\delta;\theta_0)$:

$$\forall\delta > 0, \quad C(\delta;\widehat{\theta}_n) \xrightarrow[n\to\infty]{Pr} C(\delta;\theta_0).$$

Next proposition states that the asymptotic normality also holds. It is a straightforward application of the classical δ-method, see Theorem 3.1.

Theorem 3.3 *Assume that Equation (4) is satisfied and that $\theta \mapsto C(\delta;\theta)$ is differentiable for any $\delta > 0$. Let $\nabla_\theta C(\delta;\theta)$ be the gradient vector of the cost function (with respect to θ). If $\nabla_\theta C(\delta;\theta)$ is continuous and if $\nabla_\theta C(\delta;\theta_0) \neq 0_{\mathbb{R}^p}$, then $C(\delta;\widehat{\theta}_n)$ is an asymptotic normal (point-wise) estimator of $C(\delta;\theta_0)$:*

$$\forall \delta > 0, \quad \sqrt{n}\left(C(\delta;\widehat{\theta}_n) - C(\delta;\theta_0)\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2_{cost}),$$

with $\sigma^2_{cost} = \nabla_\theta C(\delta;\theta_0)\Sigma_0 \nabla_\theta C(\delta;\theta_0)^T$.

As an application of Theorem 3.2, we can also prove that $\widehat{\delta}^\star_n$ is also an asymptotically estimator of $\delta^\star_0$ under some regularity assumptions of the function ϕ.

Theorem 3.4 *Assume that Equation (4) is satisfied and that $\theta \mapsto \phi(\delta;\theta)$ is differentiable for any $\delta > 0$. Let $\nabla_\theta \phi(\delta;\theta)$ be the gradient vector of ϕ (with respect to θ). If $\nabla_\theta\phi(\delta;\theta)$ is continuous with $\nabla_\theta\phi(\delta;\theta_0) \neq 0_{\mathbb{R}^p}$ and if $f_T(\delta^\star_0;\theta_0) \neq 0$, then $\widehat{\delta}^\star_n$ is an asymptotic normal estimator of $\delta^\star_0$:*

$$\sqrt{n}\left(\widehat{\delta}^\star_n - \delta^\star_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2_{opt.delay}),$$

where $\sigma^2_{opt.delay}$ is given by:

$$\sigma^2_{opt.delay} = \frac{\nabla_\theta\phi(\delta^\star_0;\theta_0)\Sigma_0\nabla_\theta\phi(\delta^\star_0;\theta_0)^T}{[\delta^\star_0 f_T(\delta^\star_0;\theta_0)]^2}.$$

Similarly, Theorem 3.2 can be also used to prove the asymptotic normality of the optimal delay $\widehat{C}^\star_n$.

Theorem 3.5 *Assume that Equation (4) is satisfied and that $\theta \mapsto \psi(C^\star;\theta)$ is differentiable for any $C^\star > 0$. Let $\nabla_\theta\psi(C^\star;\theta)$ be the gradient vector of ψ (with respect to θ). If $\nabla_\theta\psi(C^\star;\theta)$ is continuous with $\nabla_\theta\psi(C^\star_0;\theta_0) \neq 0_{\mathbb{R}^p}$, then $\widehat{C}^\star_n$ is an asymptotic normal estimator of $C^\star_0$:*

$$\sqrt{n}\left(\widehat{C}^\star_n - C^\star_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2_{opt.cost}),$$

where $\sigma^2_{opt.cost}$ is given by:

$$\sigma^2_{opt.cost} = \frac{c_u^2\nabla_\theta\psi(C^\star_0;\theta_0)\Sigma_0\nabla_\theta\psi(C^\star_0;\theta_0)^T}{\left[F_T^{-1}(C^\star_0/cu;\theta_0)\right]^2}.$$

It is easy to see that the gradient function (with respect to θ) involved in the above asymptotic variance can also be expressed as follows:

$$\nabla_\theta\psi(C^\star_0;\theta_0) = \int_0^{C^\star_0/c_u} \nabla_\theta F_T^{-1}(u;\theta_0)\mathrm{d}u.$$

4 Application to the case of exponentially distributed time-to-failure

We now assume that the time-to-failure T is exponentially distributed with unknown parameter $\lambda_0 \in \mathbb{R}_+ = \Theta$:

$$\forall t \geq 0, \quad S_T(t) = e^{-\lambda_0 t}.$$

Assume that we observe a n-sample $T_1,\ldots,T_n$ of time-to-failure distributed as T. The maximum likelihood estimator is given by the following natural estimator:

$$\widehat{\lambda}_n = \frac{n}{\sum_{i=1}^n T_i}.$$

It is well-known that this estimator is convergent and asymptotically normal:

$$\sqrt{n}\left(\widehat{\lambda}_n - \lambda_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \lambda_0^2).$$

For such distribution, the cost function turns to be:

$$\begin{aligned} C(\delta;\lambda_0) &= \frac{1}{\delta}\left\{c_r + c_u \int_0^\delta (1 - \exp(-\lambda_0 u))\,\mathrm{d}u\right\} \\ &= \frac{1}{\delta}\left\{c_r + c_u\delta - \frac{c_u}{\lambda_0}\left(1 - e^{-\lambda_0\delta}\right)\right\}. \end{aligned}$$

For any fixed value of δ, the partial derivative of C with respect to λ_0 is equal to :

$$\partial_\lambda C(\delta;\lambda_0) = \frac{c_u}{\delta}\left(\frac{1}{\lambda_0^2} - \left(\frac{\delta}{\lambda_0} + \frac{1}{\lambda_0^2}\right)e^{-\lambda_0\delta}\right).$$

It follows that, for any $\delta > 0$, the asymptotic variance in Theorem 3.3 is equal to:

$$\sigma^2_{cost} = c_u^2\left[\frac{1}{\lambda_0\delta} - \left(1 + \frac{1}{\lambda_0\delta}\right)e^{-\lambda_0\delta}\right]^2.$$

Now let us consider the function ϕ introduced in the previous section. For such distribution, it turns to be:

$$\phi(\delta;\lambda_0) = \frac{1}{\lambda_0} - \left(\delta + \frac{1}{\lambda_0}\right)e^{-\lambda_0\delta} - \frac{c_r}{c_u}.$$

The first order partial derivatives of ϕ are given by:

$$\partial_\delta\phi(\delta;\lambda_0) = \lambda_0\delta e^{-\lambda_0\delta}$$

and

$$\partial_\lambda\phi(\delta;\lambda_0) = -\frac{1}{\lambda_0^2} + \left(1 + \lambda_0\delta - \frac{1}{\lambda_0^2}\right)e^{-\lambda_0\delta}.$$

It follows that the asymptotic variance in Theorem 3.4 is equal to:

$$\sigma^2_{opt.delay} = \left[\frac{\delta_0^\star e^{-\lambda_0\delta_0^\star}}{(1 + \lambda_0^2 + \delta_0^\star\lambda_0^3)e^{-\lambda_0\delta_0^\star} - 1}\right]^2.$$

At least, using the expression of the quantile function for the exponential distribution, we obtain that the optimal cost $C_0^\star$ satisfies the following equation:

$$\psi(C_0^\star;\lambda_0) = \left(1 - \frac{C_0^\star}{c_u}\right)\log\left(1 - \frac{C_0^\star}{c_u}\right) + \frac{C_0^\star}{c_u} - \lambda_0\frac{c_r}{c_u} = 0.$$

Then, the first order partial derivatives of ψ are given by:

$$\partial_\delta\psi(C^\star;\lambda_0) = -\frac{1}{c_u}\log\left(1 - \frac{C_0^\star}{c_u}\right)$$

and

$$\partial_\lambda\psi(C^\star;\lambda_0) = -\frac{c_r}{c_u}.$$

It follows that the asymptotic variance in Theorem 3.5 is equal to:

$$\sigma^2_{opt.cost} = \left[\lambda_0\log\left(1 - \frac{C_0^\star}{c_u}\right)\right]^2.$$

5 Conclusion

In this paper, we have proposed a general framework for the sensitivity analysis of the block replacement policy where uncertainty comes from unknown parameters for the time-to-failure distribution that should be estimated. The case of exponentially distributed time-to-failure has been studied to illustrate the approach we have developed. Different parametric distributions can be also considered and the sensitivity analysis of some other replacement policies can be made in the same way. At least, this approach could also be used to study the sensibility of more complex policies, like, e.g., condition-based maintenance policy.

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